# A new vector coherent-state construction of SU(3) supset SO(3) Wigner coefficients



1990 J. Phys. A: Math. Gen. 23 407

(http://iopscience.iop.org/0305-4470/23/4/012)

J. Phys. A: Math. Gen. 23 (1990) 407-425. Printed in the UK

# A new vector coherent-state construction of SU(3) ⊃ SO(3) Wigner coefficients†

K T Hecht‡

Physics Department, University of Michigan, Ann Arbor, MI 48109, USA

Received 15 August 1989

**Abstract.** The new rotor coherent-state construction of the SU(3) algebra is used to derive simple expressions for SU(3) ⊃ SO(3) Wigner coefficients.

## 1. Introduction

In the past few years a vector coherent-state (VCS) theory (Rowe 1984, Rowe *et al* 1988, Hecht 1987, Deenen and Quesne 1984) has been used to great advantage to evaluate very explicit expressions for the matrix representations of many higher-rank symmetry algebras of interest in physical applications. The early detailed applications have focused on the matrix elements of the generators of the algebras. Very recently (Hecht 1989) VCS theory has been generalised to include Bargmann space realisations of more general operators lying outside the algebra, such as the operators which can generate the fundamental and other simple Wigner coefficients for these higher-rank algebras. All these applications have made use of a VCS theory based on an $n$-dimensional Bargmann variable $z$, with a scalar product defined in terms of the standard Bargmann measure. This version of the VCS theory is particularly well suited for the matrix representations of the canonical $SU(n) \supset SU(n-1)$ algebras and has led to spectacularly simple expressions for certain types of Wigner coefficients (LeBlanc and Biedenharn 1989). Some classes of SU(3) ⊃ SU(2) reduced Wigner coefficients, e.g., are simple products of SU(2) 9-$j$ coefficients and extremely simple normalisation factor ratios, the $K$-matrix ratios of VCS theory. These results indicate that VCS theory may be an important tool in simplifying the Wigner-Racah calculus for the SU(3) ⊃ SU(2) × U(1) or the more general $SU(n) \supset SU(n-1) \times U(1)$ algebras.

In another recent development in VCS theory, a new type of vector coherent state has been constructed for the SU(3) Lie algebra (Rowe *et al* 1989) to give a rotor expansion of this algebra. The new VCS state makes use of the conventional angular measure of angular momentum coherent-state theory and is well suited to calculate SU(3) matrix elements in an SO(3)-coupled basis. The new coherent state has again been used to calculate the matrix elements of the group generators, in particular the matrix elements of the SU(3) quadrupole operators, $Q_{2\mu}$, in an SU(3) ⊃ SO(3) basis of good orbital angular momentum, $L$. The new coherent state can, however, again

† Work initiated at the 7th Summer Institute in Theoretical Physics, Queen's University, Kingston, Canada, 1989.

‡ Supported in part by the US National Science Foundation.

0305-4470/90/040407 + 19$03.50 

be used to construct the rotor realisations of other operators lying outside the SU(3) algebra. The new angular coherent-state realisation of such operators is now very parallel to that of the SU(3) generators and does not require some of the special techniques used for the Bargmann space realisations of the earlier VCS constructions.

It is the purpose of the present contribution to give the new rotor realisations of operators which can generate the fundamental and other simple Wigner coefficients in an SU(3) ⊃ SO(3) coupled basis. These lead to SU(3) ⊃ SO(3) reduced Wigner coefficients in essentially analytic form. Such coefficients have been given previously (Vergados 1968) in a special orthonormalised basis, but analytic expressions were limited to SU(3) representations, $(\lambda\mu)$, with $\mu \leq 3$. A computer code (Draayer and Akiyama 1973a, b) is also available. Our main aim therefore is to demonstrate how VCS theory can be used to expedite the calculation of Wigner coefficients of higher-rank algebras. The simplicity of the final result may, however, also lead to practical computational applications in realistic nuclear physics calculations where large numbers of SU(3) ⊃ SO(3) Wigner coefficients may be required, especially when large values of the SU(3) quantum numbers $(\lambda\mu)$ come into play.

## 2. The new VCS construction of the fundamental tensors

In the new rotor coherent-state realisation of SU(3) (Rowe *et al* 1989) state vectors $|\Psi\rangle$ are mapped into their coherent-state wavefunctions $\Psi(\Omega)$ via

$$|\Psi\rangle \to \Psi(\Omega) = \langle \phi_{\lambda\mu} | R(\Omega) | \Psi \rangle \tag{1}$$

where $R(\Omega)$ is a general rotation operator, $\Omega \in \mathrm{SO}(3)$, which transforms an angular momentum eigenvector $|\alpha LM\rangle$ into

$$R(\Omega)|\alpha LM\rangle = \sum_K |\alpha LK\rangle D^L_{KM}(\Omega). \tag{2}$$

The state $|\phi_{\lambda\mu}\rangle$ is the highest-weight or intrinsic SU(3) state (Elliott 1958) with Elliott U(1)×SU(2) quantum numbers $\varepsilon = 2\lambda + \mu$, $\Lambda = M_\Lambda = \frac{1}{2}\mu$. An explicit construction of this state is well known in an SU(3)×SU(2) basis generated by oscillator creation operators, $\alpha^\dagger_{ip}$, with 'particle' index $p = 1, 2$ and $i = 1, 2, 3$ or $z, x, y$:

$$|\phi_{\lambda\mu}\rangle = N_{\lambda\mu} \alpha^{\dagger\lambda}_{z1} (\alpha^\dagger_{z1}\alpha^\dagger_{x2} - \alpha^\dagger_{x1}\alpha^\dagger_{z2})^\mu |0\rangle \tag{3}$$

with

$$N_{\lambda\mu} = \left( \frac{(\lambda+1)}{(\lambda+\mu+1)!\,\mu!} \right)^{1/2}. \tag{4}$$

Note that this is a highest-weight state in both SU(3) and SU(2), which is annihilated by both the SU(3) 'raising' generators $C_{ij}$, with $i < j$, and the SU(2) raising generator $E_{12}$, where

$$C_{ij} = \sum_{p=1}^{2} \alpha^\dagger_{ip}\alpha_{jp} \qquad i, j = z, x, y \text{ or } 1, 2, 3 \tag{5}$$

$$E_{pq} = \sum_{i=1}^{3} \alpha^\dagger_{ip}\alpha_{iq} \qquad p, q = 1, 2. \tag{6}$$

Finally, note that it is often convenient to convert the $\alpha^{\dagger}_{ip}$ to standard spherical tensor form

$$\alpha^{\dagger}_{m=\pm 1,p} = \mp \frac{1}{\sqrt{2}} (\alpha^{\dagger}_{xp} \pm \mathrm{i}\alpha^{\dagger}_{yp}) \qquad \alpha^{\dagger}_{m=0,p} = \alpha^{\dagger}_{zp}. \tag{7}$$

Under the map $|\Psi\rangle \to \Psi(\Omega)$ operators $X$ acting on $|\Psi\rangle$ are mapped into $\Gamma(X)$

$$X|\Psi\rangle \to \Gamma(X)\Psi(\Omega) = \langle \phi_{\lambda\mu}|R(\Omega)X|\Psi\rangle. \tag{8}$$

The $\Gamma(X)$ give a non-unitary realisation of the operators $X$ (with respect to the standard rotational measure $\mathrm{d}\Omega$). As in the earlier VCS constructions this is transformed to a unitary realisation $\gamma(X)$ via the similarity transformation

$$\gamma(X) = \mathcal{K}^{-1}\Gamma(X)\mathcal{K} \tag{9}$$

where the matrix elements of the $\mathcal{K}$ operator (Rowe *et al* 1988) can be evaluated (Rowe *et al* 1989) by the usual techniques (see also the appendix).

In Rowe *et al* (1989) the operators $X$ were restricted to belong to the set of SU(3) generators, which cannot change the quantum numbers $(\lambda\mu)$. If the $X$ are classified as SU(3) $\supset$ SO(3) irreducible tensors

$$X = T(X)^{(\lambda_0\mu_0)}_{lm}$$

the technique of Rowe *et al* (1989) can easily be generalised. The operator map now gives

$$\begin{aligned}\Gamma(X)^{(\lambda_0\mu_0)}_{lm}\Psi(\Omega) &= \langle \phi_{\lambda\mu}|R(\Omega)T^{(\lambda_0\mu_0)}_{lm}|\Psi\rangle \\ &= \langle \phi_{\lambda\mu}|R(\Omega)T^{(\lambda_0\mu_0)}_{lm}R^{-1}(\Omega)R(\Omega)|\Psi\rangle \\ &= \sum_{\nu} \langle \phi_{\lambda\mu}|T^{(\lambda_0\mu_0)}_{l\nu}R(\Omega)|\Psi\rangle D^{l}_{\nu m}(\Omega).\end{aligned} \tag{10}$$

The operators $T^{(\lambda_0\mu_0)}_{l\nu}$ will be constructed to be shift operators which lead to specific $(\lambda'\mu')$ values through their (left) actions on $\langle\phi_{\lambda\mu}|$. For simple SU(3) tensors $(\lambda_0\mu_0)$ this can be achieved by coupling these operators in their $p$, $q = 1$, 2-particle space to a resultant SU(2) highest-weight state of appropriate $\lambda'$ (using the technique of LeBlanc and Rowe (1986), see also Hecht (1987, ch 5)). For the fundamental tensors, e.g., with $(\lambda_0\mu_0) = (10)$, the tensors

$$T'^{(10)}_{l=1m} = \alpha^{\dagger}_{m2} \tag{11}$$

$$T''^{(10)}_{l=1m} = \frac{1}{[\lambda(\lambda+1)]^{1/2}} \{\lambda\alpha^{\dagger}_{m1} + E_{12}\alpha^{\dagger}_{m2}\} \tag{12}$$

convert the highest-weight state $\langle\phi_{\lambda\mu}|$ through their left action to states with $(\lambda'\mu') = (\lambda+1, \mu-1)$ and $(\lambda-1, \mu)$, respectively. This can be seen at once from the operator actions on the highest-weight state of (3), which gives

$$\alpha_{x2}|\phi_{\lambda\mu}\rangle = \left(\frac{\mu(\lambda+1)}{(\lambda+2)}\right)^{1/2} |\phi_{\lambda+1,\mu-1}\rangle \tag{13a}$$

$$\alpha_{z2}|\phi_{\lambda\mu}\rangle = -\left(\frac{\mu}{(\lambda+1)(\lambda+2)}\right)^{1/2} C_{xz}|\phi_{\lambda+1,\mu-1}\rangle \tag{13b}$$

$$\frac{1}{[\lambda(\lambda+1)]^{1/2}} (\lambda\alpha_{x1} + \alpha_{x2}E_{21})|\phi_{\lambda\mu}\rangle = 0 \tag{14a}$$

$$\frac{1}{[\lambda(\lambda+1)]^{1/2}} (\lambda\alpha_{z1} + \alpha_{z2}E_{21})|\phi_{\lambda\mu}\rangle = [(\lambda+\mu+1)]^{1/2}|\phi_{\lambda-1,\mu}\rangle. \tag{14b}$$

Note that it is easier to carry out the actual calculations in terms of right actions involving Hermitian conjugate operators. Note also that the operators of (12) or (14) are constructed via the $p=1$, 2-particle space SU(2) coupling. For example,

$$\frac{1}{[\lambda(\lambda+1)]^{1/2}}(\lambda\alpha_{i1}+\alpha_{i2}E_{21})|\phi_{\lambda\mu}\rangle=\sum_{m_1(m_2)} t(\alpha_i)^{1/2}_{m_2}\left|\frac{\lambda}{2}\,m_1\right\rangle\left\langle\frac{\lambda}{2}\,m_1\,\frac{1}{2}\,m_2\right|\left.\frac{\lambda-1}{2}\,\frac{\lambda-1}{2}\right\rangle \quad (15)$$

where

$$t(\alpha_i)^{1/2}_{+1/2}=-\alpha_{i2} \qquad t(\alpha_i)^{1/2}_{-1/2}=\alpha_{i1} \quad (16)$$

and

$$\left|\frac{\lambda}{2}\,\frac{\lambda}{2}-1\right\rangle=\frac{E_{21}}{\sqrt{\lambda}}\left|\frac{\lambda}{2}\,\frac{\lambda}{2}\right\rangle. \quad (17)$$

The operators $T'$ and $T''$ of (11) and (12) are the SU(3) fundamental (10)-tensors which add one square to rows 2 and 1, respectively, of the Young tableau for the SU(3) representation $(\lambda'\mu')$. Since our explicit state construction in terms of the $[\mathrm{SU}(3)\times\mathrm{SU}(2)]$ operators, $\alpha^{\dagger}_{ip}$ with $p=1,2$ only, does not permit three-rowed tableaux, the third fundamental (10)-tensor will be constructed in terms of operators which annihilate one antisymmetrically coupled pair, subtracting one square each from rows 1 and 2 (in place of an operator which would have added one square to row 3). Thus

$$T'''^{(10)}_{l=1m}=\frac{1}{\sqrt{2}}(\alpha_{m_11}\alpha_{m_22}-\alpha_{m_21}\alpha_{m_12}) \quad (18)$$

with $(mm_1m_2)$ cyclic permutations of $(+1,0,-1)$. Note that these $T'''$ convert the state $\langle\phi_{\lambda\mu}|$ through their left action to states with $(\lambda'\mu')=(\lambda,\mu+1)$. This can be seen at once from

$$(\alpha^{\dagger}_{z1}\alpha^{\dagger}_{x2}-\alpha^{\dagger}_{x1}\alpha^{\dagger}_{z2})|\phi_{\lambda\mu}\rangle=[(\lambda+\mu+2)(\mu+1)]^{1/2}|\phi_{\lambda,\mu+1}\rangle \quad (19a)$$

$$(\alpha^{\dagger}_{y1}\alpha^{\dagger}_{z2}-\alpha^{\dagger}_{z1}\alpha^{\dagger}_{y2})|\phi_{\lambda\mu}\rangle=-\left(\frac{(\lambda+\mu+2)}{(\mu+1)}\right)^{1/2}C_{yx}|\phi_{\lambda,\mu+1}\rangle \quad (19b)$$

$$(\alpha^{\dagger}_{x1}\alpha^{\dagger}_{y2}-\alpha^{\dagger}_{y1}\alpha^{\dagger}_{x2})|\phi_{\lambda\mu}\rangle=\frac{1}{[(\mu+1)(\lambda+\mu+2)]^{1/2}}O_{yz}|\phi_{\lambda,\mu+1}\rangle \quad (19c)$$

where the operator

$$O_{yz}=C_{yx}C_{xz}-C_{yz}(C_{xx}-C_{yy}+1) \quad (20)$$

is the step-down operator which converts the Elliott SU(3) intrinsic state with $\varepsilon$, $M_\Lambda=\Lambda$ to a state with $\varepsilon-3$, $\Lambda'=M'_\Lambda=\Lambda-\frac{1}{2}$.

Using the operators $T'''^{(10)}_{l=1m}$ as our prime example, we note that (10), together with (19), yields

$$\begin{aligned}\Gamma(T'''^{(10)}_{l=1m})\Psi(\Omega)&\\ =\tfrac{1}{2}\langle\phi_{\lambda\mu}|(\alpha_{z1}\alpha_{x2}-\alpha_{x1}\alpha_{z2})R(\Omega)|\Psi\rangle&(D^1_{1m}(\Omega)+D^1_{-1m}(\Omega))\\ -\frac{\mathrm{i}}{2}\langle\phi_{\lambda\mu}|(\alpha_{y1}\alpha_{z2}-\alpha_{z1}\alpha_{y2})R(\Omega)|\Psi\rangle&(D^1_{1m}(\Omega)-D^1_{-1m}(\Omega))\\ +\frac{\mathrm{i}}{\sqrt{2}}\langle\phi_{\lambda\mu}|(\alpha_{x1}\alpha_{y2}-\alpha_{y1}\alpha_{x2})R(\Omega)|\Psi\rangle&D^1_{0m}(\Omega)\end{aligned}$$

$$= \tfrac{1}{2}[(\lambda+\mu+2)(\mu+1)]^{1/2}\langle\phi_{\lambda\mu+1}|R(\Omega)|\Psi\rangle(D^1_{1m}(\Omega)+D^1_{-1m}(\Omega))$$

$$+\frac{\mathrm{i}}{2}\left(\frac{(\lambda+\mu+2)}{(\mu+1)}\right)^{1/2}\langle\phi_{\lambda\mu+1}|(C_{xy}-C_{yx})R(\Omega)|\Psi\rangle(D^1_{1m}(\Omega)-D^1_{-1m}(\Omega))$$

$$+\frac{\mathrm{i}}{\sqrt{2}}\frac{1}{[(\mu+1)(\lambda+\mu+2)]^{1/2}}\langle\phi_{\lambda\mu+1}|\{(C_{zx}-C_{xz})(C_{xy}-C_{yx})$$

$$-(\mu+2)(C_{zy}-C_{yz})\}R(\Omega)|\Psi\rangle D^1_{0m}(\Omega)$$

$$=\frac{1}{2[(\lambda+\mu+2)(\mu+1)]^{1/2}}\Bigg\{(\lambda+\mu+2)[\langle\phi_{\lambda\mu+1}|(\mu+1-L_0)R(\Omega)|\Psi\rangle D^1_{1m}(\Omega)$$

$$+\langle\phi_{\lambda\mu+1}|(\mu+1+L_0)R(\Omega)|\Psi\rangle D^1_{-1m}(\Omega)]$$

$$-\frac{1}{\sqrt{2}}\langle\phi_{\lambda\mu+1}|[L_+(L_0+\mu+2)-L_-(L_0-\mu-2)]R(\Omega)|\Psi\rangle D^1_{0m}(\Omega)\Bigg\} \qquad (21)$$

where $\langle\phi_{\lambda\mu+1}|C_{xy}$ has been converted to $\langle\phi_{\lambda\mu+1}|(C_{xy}-C_{yx})$, etc, using the properties of the highest-weight state, and where we have used

$$(C_{xy}-C_{yx})=\mathrm{i}L_0 \qquad (C_{zx}-C_{xz})=\mathrm{i}L_y=\tfrac{1}{2}(L_+-L_-)$$
$$(C_{yz}-C_{zy})=\mathrm{i}L_x=\tfrac{1}{2}\mathrm{i}(L_++L_-). \qquad (22)$$

These operators are converted to 'left-action' angular momentum operators, $\bar{L}_+$, $\bar{L}_-$, $\bar{L}_0$, via

$$\langle\phi_{\lambda'\mu'}|R(\Omega)|\Psi\rangle=\Psi_{\lambda'\mu'}(\Omega) \qquad (23a)$$

$$\langle\phi_{\lambda'\mu'}|L_0R(\Omega)|\Psi\rangle=\bar{L}_0\Psi_{(\lambda'\mu')}(\Omega) \qquad (23b)$$

$$\langle\phi_{\lambda'\mu'}|L_\pm L_0R(\Omega)|\Psi\rangle=\bar{L}_\pm\bar{L}_0\Psi_{\lambda'\mu'}(\Omega). \qquad (23c)$$

The actions of the operators $\bar{L}_k$ are defined by expanding $|\Psi\rangle$ in terms of angular momentum eigenvectors $|\alpha'L'M'\rangle$. Consider, for example,

$$\langle\phi_{\lambda'\mu'}|L_0R(\Omega)|\alpha'L'M'\rangle=\sum_{K'}\langle\phi_{\lambda'\mu'}|L_0|\alpha'L'K'\rangle D^{L'}_{K'M'}(\Omega) \qquad (24a)$$

or

$$\langle\phi_{\lambda'\mu'}|L_\pm L_0R(\Omega)|\alpha'L'M'\rangle=\sum_{K'}\langle\phi_{\lambda'\mu'}|L_\pm L_0|\alpha'L'K'\rangle D^{L'}_{K'M'}(\Omega). \qquad (24b)$$

From these we infer that

$$\bar{L}_0D^{L'}_{K'M'}(\Omega)=K'D^{L'}_{K'M'}(\Omega) \qquad (25a)$$

$$\bar{L}_\pm\bar{L}_0D^{L'}_{K'M'}(\Omega)=[(L'\pm K')(L'\mp K'+1)]^{1/2}(K'\mp 1)D^{L'}_{K'\mp1,M'}(\Omega). \qquad (25b)$$

Finally, with the use of the standard rotational measure

$$\frac{[(2L'+1)(2L+1)]^{1/2}}{8\pi^2}\int \mathrm{d}\Omega \qquad (26)$$

and standard $D$-function integrals, we obtain

$$\langle(\lambda\mu)KLM|\Gamma(T'''^{(10)}_{l=1m})|(\lambda'\mu')=(\lambda,\mu+1)K'L'M'\rangle$$

$$=\langle L'M'1m|LM\rangle\langle(\lambda\mu)KL\|\Gamma(T'''^{(10)}_{l=1})\|(\lambda'\mu')K'L'\rangle$$

$$=\frac{1}{2}\left(\frac{(2L'+1)}{(2L+1)}\right)^{1/2}\frac{1}{[(\lambda+\mu+2)(\mu+1)]^{1/2}}\langle L'M'1m|LM\rangle$$

$$\times\Big\{(\lambda+\mu+2)[(\mu+1-K')\langle L'K'11|LK\rangle+(\mu+1+K')\langle L'K'1-1|LK\rangle]$$

$$-\frac{1}{\sqrt{2}}(\mu+1+K')[(L'+K')(L'-K'+1)]^{1/2}\langle L'(K'-1)10|LK\rangle$$

$$-\frac{1}{\sqrt{2}}(\mu+1-K')[(L'-K')(L'+K'+1)]^{1/2}\langle L'(K'+1)10|LK\rangle\Big\} \tag{27}$$

for the special case $(\lambda'\mu')=(\lambda,\mu+1)$, with similar expressions for the other $(\lambda'\mu')$. These matrix elements of the non-unitary operator realisation $\Gamma(T)$ are now converted to those of the unitary realisation $\gamma(T)$ of (9) via the matrix elements $(\mathcal{K})_{Ki}$ (see the appendix).

Finally, the matrix elements of the unitary $\gamma(T)$ between orthonormal states $|(\lambda\mu)iLM\rangle$ can be converted to standard Hilbert space and lead to

$$\langle(\lambda\mu)iLM|\gamma(T^{(10)}_{l=1m})|(\lambda'\mu')i'L'M'\rangle$$

$$=\langle L'M'1m|LM\rangle\langle(\lambda'\mu')i'L';(10)1\|(\lambda\mu)iL\rangle\langle\langle(\lambda\mu)\|T^{(10)}\|(\lambda'\mu')\rangle\rangle$$

$$=\sum_{K,K'}(\mathcal{K}^{-1}((\lambda\mu)L))_{iK}(\mathcal{K}(\lambda'\mu')L')_{K'i'}$$

$$\times\langle L'M'1m|LM\rangle\langle(\lambda\mu)KL\|\Gamma(T^{(10)}_{l=1})\|(\lambda'\mu')K'L'\rangle. \tag{28}$$

Here, the left-hand side has been expressed in terms of an SU(3)-reduced matrix element of the operator $T^{(10)}$, denoted by double verticals and double angle brackets, and otherwise yields the SU(3) $\supset$ SO(3) reduced Wigner coefficient which is sought. The quantum numbers, $i$, of the orthonormal basis are associated with the $i$th non-zero eigenvalue of the Hermitian matrix $(\mathcal{K}\mathcal{K}^{\dagger})_{K_1K_2}$ and are most simply labelled by $i=1, 2, \ldots$. The double angle bracket reduced matrix element is dependent on the parities of $\lambda$, $\mu$, $\lambda'$ and $\mu'$ since the coherent-state constructions are based on the unique starting states with $L=0$ for $\lambda$ and $\mu$ both even or with $L=1$ for all other $\lambda$, $\mu$ combinations. For these starting states the $1\times1$ $\mathcal{K}$ matrices are set equal to unity (see Rowe *et al* 1989). It is therefore most efficient to combine the double angle bracket reduced matrix elements with the simple $\lambda$, $\mu$-dependent factors of equations such as (27), where we set

$$\frac{1}{2[(\lambda+\mu+2)(\mu+1)]^{1/2}}\frac{1}{\langle\langle(\lambda,\mu+1)\|T^{(10)}\|(\lambda\mu)\rangle\rangle}=N^{(\lambda'\mu')=(\lambda,\mu+1)}_{(\lambda\mu)} \tag{29}$$

and determine the normalisation factors $N$ from the orthonormality of the Wigner coefficients with the simplest starting values such as $L=0$ for $\lambda$ and $\mu$ both even, or $L=1$ otherwise. Finally, it should be noted that the $K$ values in (27) can be both positive and negative, whereas the $K$ values which label the $(\mathcal{K}\mathcal{K}^{\dagger})$ matrices are restricted to be positive, with $K = \mu, \mu - 2, \ldots, 0(1)$ for $\mu$ = even (odd). The SU(3) rotor states for the representation $(\lambda\mu)$ are given by the combination

$$\Psi_{KLM}(\Omega) = \left(\frac{(2L+1)}{16\pi^2(1+\delta_{K0})}\right)^{1/2} [D^L_{KM}(\Omega) + (-1)^{\lambda+L+K} D^L_{-KM}(\Omega)] \quad (30)$$

with $K$ carrying positive values only, and $(-1)^K = (-1)^\mu$. (Note that $\lambda + L$ must be even for $K = 0$.) In terms of these positive $K$ values the SU(3) ⊃ SO(3) reduced Wigner coefficients can then be expressed by

$$\begin{aligned} &\langle(\lambda'\mu')i'L'; (10)l = 1\|(\lambda\mu)iL\rangle \\ &\quad = \sum_{K,K'} (\mathcal{K}^{-1}((\lambda\mu)L))_{iK} (\mathcal{K}((\lambda'\mu')L'))_{K'i'} N^{(\lambda'\mu')}_{(\lambda\mu)} \left(\frac{(2L'+1)}{(2L+1)}\right)^{1/2} \\ &\quad \times \langle(\lambda'\mu')K'L'; (10)1\|(\lambda\mu)KL\rangle. \end{aligned} \quad (31)$$

For $(\lambda'\mu') = (\lambda, \mu+1)$ and $(\lambda+1, \mu-1)$ the $K$-dependent factors have the simple values

$$\begin{aligned} &\langle(\lambda'\mu')K'L'; (10)1\|(\lambda\mu)K = K' \pm 1, L\rangle \\ &\quad = \langle L'K'1 \pm 1|LK = K' \pm 1\rangle \begin{Bmatrix} F_{LL'}(+K')[(1+\delta_{K'0})]^{1/2} \\ (-1)^{\lambda'-\lambda} F_{LL'}(-K')[(1+\delta_{K0})]^{1/2} \end{Bmatrix} \end{aligned} \quad (32a)$$

where, for $(\lambda'\mu') = (\lambda, \mu+1)$,

$$\begin{aligned} F_{LL'}(K') &= (\mu+1-K')(\lambda+\mu+2-L'+K') && \text{for } L = L'+1 \\ &= (\mu+1-K')(\lambda+\mu+3+K') && \text{for } L = L' \\ &= (\mu+1-K')(\lambda+\mu+3+L'+K') && \text{for } L = L'-1 \end{aligned} \quad (32b)$$

and for $(\lambda'\mu') = (\lambda+1, \mu-1)$:

$$\begin{aligned} F_{LL'}(K') &= -(\lambda+1-L'+K') && \text{for } L = L'+1 \\ &= -(\lambda+K'+2) && \text{for } L = L' \\ &= -(\lambda+2+L'+K') && \text{for } L = L'-1. \end{aligned} \quad (32c)$$

Finally, for $(\lambda'\mu') = (\lambda-1, \mu)$,

$$\langle(\lambda-1, \mu)K'L'; (10)1\|(\lambda\mu)KL\rangle = \langle L'K'10|LK' = K\rangle. \quad (33)$$

The fundamental SU(3) ⊃ SO(3) Wigner coefficients have thus been expressed in terms of ordinary SU(2) Wigner coefficients and a few simple factors. The $\mathcal{K}$-matrix elements follow from the $\mathcal{K}\mathcal{K}^+$ matrices. Analytical expressions for some of the simpler cases are given in the appendix. The normalisation factors, $N$, needed for the coupling $(\lambda'\mu') \times (10) \to (\lambda\mu)$ are given in table 1. We note that the SU(3) ⊃ SO(3) Wigner coefficients of (31)–(33) are in a form very different from those given earlier by Le Blanc and Rowe (1985). The earlier expressions involve products of SU(2) Racah coefficients and require a sum over angular momentum quantum numbers.

**Table 1.** The normalisation factors $N^{(\lambda'\mu')}_{(\lambda\mu)}$ for $(\lambda'\mu')\times(10)\to(\lambda\mu)$.

| | $(\lambda'\mu')$ | | |
|---|---|---|---|
| $(\lambda\mu)$ | $(\lambda-1,\mu)$ | $(\lambda,\mu+1)$ | $(\lambda+1,\mu-1)$ |
| ee | 1 | $\frac{1}{\sqrt{2}}\frac{1}{(\mu+2)(\lambda+\mu+3)}$ | $\frac{1}{(\lambda+2)\sqrt{2}}$ |
| eo | $\left(\frac{\lambda+\mu+2}{\lambda+\mu+1}\right)^{1/2}$ | $\frac{1}{[2(\mu+1)(\mu+2)(\lambda+\mu+2)(\lambda+\mu+3)]^{1/2}}$ | $\frac{1}{(\lambda+2)}\left(\frac{\mu+1}{2\mu}\right)^{1/2}$ |
| oe | $\left(\frac{(\lambda+1)(\lambda+\mu+2)}{\lambda(\lambda+\mu+1)}\right)^{1/2}$ | $\frac{1}{(\mu+2)}\frac{1}{[2(\lambda+\mu+2)(\lambda+\mu+3)]^{1/2}}$ | $\frac{1}{[2(\lambda+1)(\lambda+2)]^{1/2}}$ |
| oo | $\left(\frac{\lambda+1}{\lambda}\right)^{1/2}$ | $\frac{1}{(\lambda+\mu+3)[2(\mu+1)(\mu+2)]^{1/2}}$ | $\left(\frac{(\mu+1)}{2\mu(\lambda+1)(\lambda+2)}\right)^{1/2}$ |

## 3. SU(3) ⊃ SO(3) Wigner coefficients for the coupling $(\lambda'\mu')\times(20)\to(\lambda\mu)$

The techniques used for the fundamental (10)-tensors in section 2 can be generalised to yield SU(3) ⊃ SO(3) Wigner coefficients for other simple couplings. This will be illustrated in this section for the SU(3) coupling $(\lambda'\mu')\times(20)\to(\lambda\mu)$. The six SU(3) (20)-tensors which induce specific shifts are constructed by the techniques of section 2 in terms of the oscillator creation and annihilation operators $\alpha^{\dagger}_{mp}(\alpha_{mp})$ with $p=1$ and 2. With the definition

$$t^{(20)}_{lm}(p,q)=\frac{1}{[(1+\delta_{pq})]^{1/2}}\sum_{m_1m_2}\langle 1m_1 1m_2|lm\rangle\alpha^{\dagger}_{m_1p}\alpha^{\dagger}_{m_2q} \tag{34}$$

the six (20) shift tensors are given by (with $l=0,2$)

for $(\lambda'\mu')=(\lambda+2,\mu-2)$: $t^{(20)}_{lm}(2,2)$

for $(\lambda'\mu')=(\lambda,\mu-1)$:

$$-\left(\frac{\lambda}{\lambda+2}\right)^{1/2}\left\{t^{(20)}_{lm}(1,2)+\frac{\sqrt{2}}{\lambda}E_{12}t^{(20)}_{lm}(2,2)\right\}$$

for $(\lambda'\mu')=(\lambda-2,\mu)$:

$$\left(\frac{\lambda-1}{\lambda+1}\right)^{1/2}\left\{t^{(20)}_{lm}(1,1)+\frac{\sqrt{2}}{\lambda}E_{12}t^{(20)}_{lm}(1,2)+\frac{1}{\lambda(\lambda-1)}E^2_{12}t^{(20)}_{lm}(2,2)\right\}$$

for $(\lambda'\mu)=(\lambda+1,\mu)$: (35)

$$\sum_{m_1m_2}\alpha^{\dagger}_{m_12}T'''^{(10)}_{l=1m_2}\langle 1m_1 1m_2|lm\rangle$$

for $(\lambda'\mu')=(\lambda-1,\mu+1)$:

$$\frac{1}{[\lambda(\lambda+1)]^{1/2}}\sum_{m_1m_2}(\lambda\alpha^{\dagger}_{m_11}+E_{12}\alpha^{\dagger}_{m_12})T'''^{(10)}_{l=1m_2}\langle 1m_1, 1m_2|lm\rangle$$

for $(\lambda'\mu')=(\lambda,\mu+2)$:

$$\frac{1}{\sqrt{3}}\sum_{m_1m_2}T'''^{(10)}_{1m_1}T'''^{(10)}_{1m_2}\langle 1m_1 1m_2|lm\rangle$$

where the $T_{1m}^{\prime\prime\prime(10)}$ are defined by (18). The action of the six shift tensors on the representations $(\lambda'\mu')$ indicated converts these to the representation $(\lambda,\mu)$ by the shifts which correspond respectively to the additions of two squares to row 2, one square each to rows 1 and 2, two squares to row 1, one square each to rows 2 and 3, one square each to rows 1 and 3, and two squares to row 3 of the Young tableau which characterises the representation $(\lambda'\mu')$. The six shift tensors thus convert the state $\langle\phi_{\lambda\mu}|$ through their left action to states with the indicated $(\lambda'\mu')$. This can again be seen at once from relations such as

$$\begin{aligned}
\alpha_{x2}^2|\phi_{\lambda\mu}\rangle &= \left(\frac{\mu(\mu-1)(\lambda+1)}{(\lambda+3)}\right)^{1/2}|\phi_{\lambda+2,\mu-2}\rangle \\
\alpha_{x2}\alpha_{z2}|\phi_{\lambda\mu}\rangle &= -\frac{1}{(\lambda+2)}\left(\frac{\mu(\mu-1)(\lambda+1)}{(\lambda+3)}\right)^{1/2}C_{xz}|\phi_{\lambda+2,\mu-2}\rangle \\
\alpha_{z2}^2|\phi_{\lambda\mu}\rangle &= \frac{1}{(\lambda+2)}\left(\frac{\mu(\mu-1)}{(\lambda+1)(\lambda+3)}\right)^{1/2}C_{xz}^2|\phi_{\lambda+2,\mu-2}\rangle
\end{aligned}\tag{36}$$

for the first case; or, as a second example,

$$\begin{aligned}
&(\alpha_{z1}^\dagger\alpha_{x2}^\dagger-\alpha_{x1}^\dagger\alpha_{z2}^\dagger)^2|\phi_{\lambda\mu}\rangle = [(\mu+1)(\mu+2)(\lambda+\mu+2)(\lambda+\mu+3)]^{1/2}|\phi_{\lambda,\mu+2}\rangle \\
&(\alpha_{z1}^\dagger\alpha_{x2}^\dagger-\alpha_{x1}^\dagger\alpha_{z2}^\dagger)(\alpha_{y1}^\dagger\alpha_{z2}^\dagger-\alpha_{z1}^\dagger\alpha_{y2}^\dagger)|\phi_{\lambda\mu}\rangle \\
&\qquad = -\left(\frac{(\mu+1)(\lambda+\mu+2)(\lambda+\mu+3)}{(\mu+2)}\right)^{1/2}C_{yx}|\phi_{\lambda,\mu+2}\rangle \\
&(\alpha_{y1}^\dagger\alpha_{z2}^\dagger-\alpha_{z1}^\dagger\alpha_{y2}^\dagger)^2|\phi_{\lambda\mu}\rangle = \left(\frac{(\lambda+\mu+2)(\lambda+\mu+3)}{(\mu+1)(\mu+2)}\right)^{1/2}C_{yx}^2|\phi_{\lambda,\mu+2}\rangle \\
&(\alpha_{z1}^\dagger\alpha_{x2}^\dagger-\alpha_{x1}^\dagger\alpha_{z2}^\dagger)(\alpha_{x1}^\dagger\alpha_{y2}^\dagger-\alpha_{y1}^\dagger\alpha_{x2}^\dagger)|\phi_{\lambda\mu}\rangle = \left(\frac{(\mu+1)(\lambda+\mu+2)}{(\mu+2)(\lambda+\mu+3)}\right)^{1/2}O_{yz}|\phi_{\lambda,\mu+2}\rangle \\
&(\alpha_{y1}^\dagger\alpha_{z2}^\dagger-\alpha_{z1}^\dagger\alpha_{y2}^\dagger)(\alpha_{x1}^\dagger\alpha_{y2}^\dagger-\alpha_{y1}^\dagger\alpha_{x2}^\dagger)|\phi_{\lambda\mu}\rangle \\
&\qquad = -\left(\frac{(\lambda+\mu+2)}{(\mu+1)(\mu+2)(\lambda+\mu+3)}\right)^{1/2}C_{yx}O_{yz}|\phi_{\lambda,\mu+2}\rangle \\
&(\alpha_{x1}^\dagger\alpha_{y2}^\dagger-\alpha_{y1}^\dagger\alpha_{x2}^\dagger)^2|\phi_{\lambda\mu}\rangle = \frac{1}{[(\mu+1)(\mu+2)(\lambda+\mu+2)(\lambda+\mu+3)]^{1/2}}(O_{yz})^2|\phi_{\lambda,\mu+2}\rangle
\end{aligned}\tag{37}$$

where the operator $O_{yz}$ is defined in (20). The operators which occur in such relations are converted to left-action angular momentum operators via relations such as

$$\begin{aligned}
&\langle\phi_{\lambda,\mu+2}|O_{yz}^\dagger C_{xy} \\
&\quad = \langle\phi_{\lambda,\mu+2}|(C_{zx}-C_{xz})[(C_{xy}-C_{yx})(C_{xy}-C_{yx})+(C_{xx}-C_{yy})] \\
&\qquad -\langle\phi_{\lambda,\mu+2}|(\mu+3)[(C_{zy}-C_{yz})(C_{xy}-C_{yx})+(C_{zx}-C_{xz})] \\
&\quad = \langle\phi_{\lambda,\mu+2}|\{(C_{zx}-C_{xz})(C_{xy}-C_{yx})^2-(\mu+3)(C_{zy}-C_{yz})(C_{xy}-C_{yx})\} \\
&\quad = -\langle\phi_{\lambda,\mu+2}|\tfrac{1}{2}\{(L_+-L_-)L_0^2+(\mu+3)(L_++L_-)L_0\}
\end{aligned}\tag{38}$$

or

$$\begin{aligned}
&\langle\phi_{\lambda,\mu+2}|(O_{y,z}^\dagger)^2 \\
&\quad = -\tfrac{1}{4}\langle\phi_{\lambda,\mu+2}|L_+^2\{L_0^2+(\mu+2)(\mu+4)+2(\mu+3)L_0\} \\
&\qquad -\tfrac{1}{4}\langle\phi_{\lambda,\mu+2}|L_-^2\{L_0^2+(\mu+2)(\mu+4)-2(\mu+3)L_0\} \\
&\qquad +\langle\phi_{\lambda,\mu+2}|\{(\mu+2)^2-L_0^2\}\{(\lambda+\mu+3)+\tfrac{1}{2}(L_0^2-\boldsymbol{L}\cdot\boldsymbol{L})\}.
\end{aligned}\tag{39}$$

The techniques of section 2 then lead to the SU(3) ⊃ SO(3) reduced Wigner coefficients which again have the general form:

$$\langle(\lambda'\mu')i'L';(20)l\|(\lambda\mu)iL\rangle = \sum_{K,K'}(\mathcal{K}^{-1}((\lambda\mu)L))_{iK}(\mathcal{K}((\lambda'\mu')L'))_{K'i'}N^{(\lambda'\mu')}_{(\lambda\mu)}\left(\frac{(2L'+1)}{(2L+1)}\right)^{1/2} \times\langle(\lambda'\mu')K'L';(20)l\|(\lambda\mu)KL\rangle. \quad (40)$$

The normalisation factors, $N$, are given in table 2. The $K'$, $K$-dependent factors are given below.

*3.1. For $(\lambda'\mu')=(\lambda-2,\mu)$*

$$\langle(\lambda-2,\mu)K'L';(20)2\|(\lambda\mu)KL\rangle=\langle L'K'20|LK\rangle$$
$$\langle(\lambda-2,\mu)K'L';(20)0\|(\lambda\mu)KL\rangle=\frac{1}{\sqrt{2}}\delta_{KK'}\delta_{LL'} \quad (41)$$

*3.2. For the three cases $(\lambda'\mu')=(\lambda+2,\mu-2)$, $(\lambda+1,\mu)$, $(\lambda,\mu+2)$*

$$\langle(\lambda'\mu')K'L';(20)2\|(\lambda\mu)KL\rangle = F_{L'L}(K')\langle L'K'22|LK\rangle\delta_{K,K'+2}[(1+\delta_{K'0})]^{1/2}+(-1)^{\lambda'-\lambda}F_{L'L}(-K') \times\langle L'K'2-2|LK\rangle\delta_{K,K'-2}[(1+\delta_{K0})]^{1/2}+G_{L'L}(K')(\tfrac{2}{3})^{1/2}\langle L'K'20|LK\rangle\delta_{KK'} +\delta_{KK'}\delta_{K1}F_{L'L}(-1)(-1)^{\lambda'+L'+1}\langle L'-1\,22|L1\rangle \quad (42)$$

where the factors $F_{L'L}(K')$, $G_{LL}(K')$ are given in tables 3–5. The remaining coefficients are given by

$$\langle(\lambda+2,\mu-2)K'L;(20)l=0\|(\lambda\mu)KL\rangle = \frac{2}{\sqrt{3}}\{[(\lambda+2)^2-\tfrac{1}{2}L(L+1)+\tfrac{1}{2}K'^2+\tfrac{1}{4}(-1)^\lambda L(L+1)\delta_{K1}]\delta_{KK'} +\tfrac{1}{4}[(L+K'+1)(L+K'+2)(L-K')(L-K'-1))^{1/2} \times\delta_{K,K'+2}[(1+\delta_{K'0})]^{1/2} +\tfrac{1}{4}[(L-K'+1)(L-K'+2)(L+K')(L+K'-1)]^{1/2} \times\delta_{K,K'-2}[(1+\delta_{K0})]^{1/2}\} \quad (43a)$$

$$\langle(\lambda+1,\mu)K'L;(20)0\|(\lambda\mu)KL\rangle = \frac{2}{\sqrt{3}}\{[(\lambda+1)(\lambda+\mu+3)+\tfrac{1}{2}(\mu+2)-\tfrac{1}{2}L(L+1)+\tfrac{1}{2}K'^2 +\tfrac{1}{4}(-1)^{\lambda+L+1}(\mu+1)L(L+1)\delta_{K1}]K'\delta_{KK'} +\tfrac{1}{4}(K'-\mu)[(L+K'+1)(L+K'+2)(L-K')(L-K'-1)]^{1/2} \times\delta_{K,K'+2}[(1+\delta_{K'0})]^{1/2} +\tfrac{1}{4}(K'+\mu)[(L-K'+1)(L-K'+2)(L+K')(L+K'-1)]^{1/2} \times\delta_{K,K'-2}[(1+\delta_{K0})]^{1/2}\} \quad (43b)$$

**Table 2.** The normalisation factors $N_{(\lambda\mu)}^{(\lambda'\mu')}$ for $(\lambda'\mu')\times(20)\to(\lambda\mu)$.

| | $(\lambda'\mu')$ | | |
|---|---|---|---|
| $(\lambda\mu)$ | $(\lambda+2,\mu-2)$ | $(\lambda-2,\mu)$ | $(\lambda,\mu+2)$ |
| ee | $\frac{1}{2(\lambda+2)}\left(\frac{\mu}{(\mu-1)(\lambda+2)(\lambda+3)}\right)^{1/2}$ | $\left(\frac{2\lambda(\lambda+\mu+1)}{3(\lambda-1)(\lambda+\mu)}\right)^{1/2}$ | $\frac{1}{(\lambda+\mu+3)(\mu+2)2[(\mu+2)(\mu+3)(\lambda+\mu+3)(\lambda+\mu+4)]^{1/2}}$ |
| eo | $\frac{1}{2(\lambda+2)}\left(\frac{\mu+1}{\mu(\lambda+2)(\lambda+3)}\right)^{1/2}$ | $\left(\frac{2\lambda(\lambda+\mu+2)}{3(\lambda-1)(\lambda+\mu+1)}\right)^{1/2}$ | $\frac{1}{(\mu+3)(\lambda+\mu+4)2[(\mu+1)(\mu+2)(\lambda+\mu+2)(\lambda+\mu+3)]^{1/2}}$ |
| oe | $\frac{1}{2(\lambda+3)}\left(\frac{\mu}{(\mu-1)(\lambda+1)(\lambda+2)}\right)^{1/2}$ | $\left(\frac{2(\lambda+1)(\lambda+\mu+2)}{3\lambda(\lambda+\mu+1)}\right)^{1/2}$ | $\frac{1}{(\mu+2)(\lambda+\mu+4)2[(\mu+3)(\mu+2)(\lambda+\mu+2)(\lambda+\mu+3)]^{1/2}}$ |
| oo | $\frac{1}{2(\lambda+3)}\left(\frac{\mu}{(\mu-1)(\lambda+1)(\lambda+2)}\right)^{1/2}$ | $\left(\frac{2(\lambda+1)(\lambda+\mu+1)}{3\lambda(\lambda+\mu)}\right)^{1/2}$ | $\frac{1}{(\mu+3)(\lambda+\mu+3)2[(\mu+1)(\mu+2)(\lambda+\mu+3)(\lambda+\mu+4)]^{1/2}}$ |
| $(\lambda\mu)$ | $(\lambda-1,\mu+1)$ | $(\lambda,\mu-1)$ | $(\lambda+1,\mu)$ |
| ee | $\frac{-1}{(\mu+2)[2(\lambda+\mu+1)(\lambda+\mu+3)]^{1/2}}$ | $-\frac{1}{[2\lambda(\lambda+2)]^{1/2}}$ | $\frac{-1}{(\lambda+2)(\lambda+\mu+3)[2\mu(\mu+2)]^{1/2}}$ |
| eo | $\frac{1}{[2(\mu+1)(\mu+2)(\lambda+\mu+1)(\lambda+\mu+3)]^{1/2}}$ | $-\left(\frac{(\mu+1)(\lambda+\mu+2)}{2\mu\lambda(\lambda+2)(\lambda+\mu+1)}\right)^{1/2}$ | $\frac{-1}{(\lambda+2)[2\mu(\mu+2)(\lambda+\mu+2)(\lambda+\mu+3)]^{1/2}}$ |
| oe | $\frac{1}{(\mu+2)}\left(\frac{(\lambda+1)}{2\lambda(\lambda+\mu+1)(\lambda+\mu+3)}\right)^{1/2}$ | $\left(\frac{(\lambda+\mu+2)}{2\lambda(\lambda+2)(\lambda+\mu+1)}\right)^{1/2}$ | $\frac{-1}{[2(\lambda+1)(\lambda+2)\mu(\mu+2)(\lambda+\mu+2)(\lambda+\mu+3)]^{1/2}}$ |
| oo | $\left(\frac{(\lambda+1)}{2\lambda(\mu+2)(\mu+1)(\lambda+\mu+1)(\lambda+\mu+3)}\right)^{1/2}$ | $-\left(\frac{(\mu+1)}{2\mu\lambda(\lambda+2)}\right)^{1/2}$ | $\frac{-1}{(\lambda+\mu+3)[2(\lambda+1)(\lambda+2)\mu(\mu+2)]^{1/2}}$ |

**Table 3.** The $F_{L'L}(K')$, $G_{L'L}(K')$ for $(\lambda'\mu') = (\lambda+2, \mu-2)$.

| $L$ | $F_{L'L}(K')$ | $G_{L'L}(K')$ |
|---|---|---|
| $L'+2$ | $(\lambda+1-L'+K')(\lambda+2-L'+K')$ | $-[(\lambda-1-L')(\lambda+2-L')-K'^2]$ |
| $L'+1$ | $(\lambda+2-L'+K')(\lambda+3+K')$ | $-[\lambda(\lambda+3-L')+L'^2-K'^2]$ |
| $L'$ | $(\lambda+3+K')^2-\frac{1}{3}L'(L'+1)$ | $-[\lambda^2+4\lambda+1+L'(L'+1)-K'^2]$ |
| $L'-1$ | $(\lambda+3+L'+K')(\lambda+3+K')$ | $-[\lambda^2+4\lambda+1+(\lambda+2)L'+L'^2-K'^2]$ |
| $L'-2$ | $(\lambda+2+L'+K')(\lambda+3+L'+K')$ | $-[(\lambda+L')(\lambda+3+L')-K'^2]$ |

**Table 4.** The $F_{L'L}(K')$, $G_{L'L}(K')$ for $(\lambda'\mu') = (\lambda+1, \mu)$.

$$F_{L'L}(K') = (K'-\mu)\{(\lambda+1)(\lambda+\mu+2)+f_{L'L}(K')\}$$

| $L$ | $f_{L'L}(K')$ |
|---|---|
| $L'+2$ | $(L'-K')(L'-K'-3-2\lambda-\mu)$ |
| $L'+1$ | $-(K'+2)(L'-K'-1)-\frac{1}{2}(2\lambda+\mu+2)(L'-2K'-2)$ |
| $L'$ | $(K'+2)^2-\frac{1}{3}L'(L'+1)+\frac{1}{2}(2\lambda+\mu+2)(2K'+3)$ |
| $L'-1$ | $(K'+2)(L'+K'+2)+\frac{1}{2}(2\lambda+\mu+2)(L'+2K'+3)$ |
| $L'-2$ | $(L'+K'+1)(L'+K'+4+2\lambda+\mu)$ |

$$G_{L'L}(K') = K'\{-(\lambda+1)(\lambda+\mu)+(\mu+2)-L'(L'+1)+K'^2+g_{L'L}(K')\}$$

| $L$ | $g_{L'L}(K')$ |
|---|---|
| $L'+2$ | $\{L'(2\lambda+\mu+2)-\mu(\mu+2)\}$ |
| $L'+1$ | $\frac{1}{2K'^2}\{K'^2(2\lambda+\mu+2)(L'-2)+\mu(\mu+2)[L'(L'+2)-2K'^2]\}$ |
| $L'$ | $\frac{3}{2}\frac{1}{[3K'^2-L'(L'+1)]}\{(2\lambda+\mu+2)[L'(L'+1)-3K'^2]+\mu(\mu+2)[2L'(L'+1)1-2K'^2]\}$ |
| $L'-1$ | $-\frac{1}{2K'^2}\{K'^2(2\lambda+\mu+2)(L'+3)-\mu(\mu+2)[(L'-1)(L'+1)-2K'^2]\}$ |
| $L'-2$ | $-\{(2\lambda+\mu+2)(L'+1)+\mu(\mu+2)\}$ |

$$\begin{aligned}
&\langle(\lambda, \mu+2)K'L; (20)0\|(\lambda\mu)KL\rangle \\
&\quad = \frac{2}{\sqrt{3}}\{[(\mu+2+K')(\mu+2-K')][(\lambda+\mu+3)^2-\tfrac{1}{2}L(L+1)+\tfrac{1}{2}K'^2 \\
&\qquad +\tfrac{1}{4}(-1)^{\lambda+L}L(L+1)(\mu+3)(\mu+1)\delta_{K1}]\delta_{KK'}-\tfrac{1}{4}(K'-\mu)(K'-\mu-2) \\
&\qquad \times[(L+K'+1)(L+K'+2)(L-K')(L-K'-1)]^{1/2}\delta_{K,K'+2}[(1+\delta_{K'0})]^{1/2} \\
&\qquad -\tfrac{1}{4}(K'+\mu)(K'+\mu+2)[(L-K'+1)(L-K'+2)(L+K')(L+K'-1)]^{1/2} \\
&\qquad \times\delta_{K,K'-2}[(1+\delta_{K0})]^{1/2}\}. \qquad (43c)
\end{aligned}$$

*3.3. For the remaining two cases $(\lambda'\mu') = (\lambda, \mu-1)$, $(\lambda-1, \mu+1)$*

$$\begin{aligned}
&\langle(\lambda'\mu')K'L; (20)0\|(\lambda\mu)KL\rangle \\
&\quad = (\tfrac{1}{3})^{1/2}[(L-K')(L+K'+1)]^{1/2}\delta_{K,K'+1}[(1+\delta_{K'0})]^{1/2}h(K') \\
&\qquad +(-1)^{\lambda'-\lambda-1}(\tfrac{1}{3})^{1/2}[(L+K')(L-K'+1)]^{1/2}\delta_{K,K'-1}[(1+\delta_{K0})]^{1/2}h(-K')
\end{aligned}$$

**Table 5.** The $F_{L'L}(K')$, $G_{L'L}(K')$ for $(\lambda'\mu')=(\lambda, \mu+2)$.

$F_{L'L}(K')=\{(\lambda+\mu+2)(K'-\mu)(K'-\mu-2)f_{L'L}(K')+(K'+\mu+4)(K'+\mu+6)\phi_{L'L}(K')\}$

| $L$ | $f_{L'L}(K')$ | $\phi_{L'L}(K')$ |
|---|---|---|
| $L'+2$ | $(\lambda+\mu+3-2L'+2K')$ | $(L'-K')(L'-K'-1)$ |
| $L'+1$ | $(\lambda+\mu+5-L'+2K')$ | $-(K'+2)(L'-K'-1)$ |
| $L'$ | $(\lambda+\mu+6+2K')$ | $\{(K'+2)^2-\frac{1}{3}L'(L'+1)\}$ |
| $L'-1$ | $(\lambda+\mu+6+L'+2K')$ | $(K'+2)(L'+K'+2)$ |
| $L'-2$ | $(\lambda+\mu+5+2L'+2K')$ | $(L'+K'+1)(L'+K'+2)$ |

$G_{L'L}(K')=(\mu+2+K')(\mu+2-K')\{(\lambda+\mu+3)(\lambda+\mu)+g_{L'L}(K')\}$

| $L$ | $g_{L'L}(K')$ |
|---|---|
| $L'+2$ | $\{L'(L'+1)-K'^2-2L'(\lambda+\mu+2)\}$ |
| $L'+1$ | $\{L'(L'+1)-K'^2-(L'-2)(\lambda+\mu+2)\}$ |
| $L'$ | $\{L'(L'+1)-K'^2+3(\lambda+\mu+2)\}$ |
| $L'-1$ | $\{L'(L'+1)-K'^2+(L'+3)(\lambda+\mu+2)\}$ |
| $L'-2$ | $\{L'(L'+1)-K'^2+2(L'+1)(\lambda+\mu+2)\}$ |

with

$$h(K')=1 \qquad \text{for } (\lambda'\mu')=(\lambda, \mu-1) \tag{44}$$

and

$$h(K')=(\mu+1-K') \qquad \text{for } (\lambda'\mu')=(\lambda-1, \mu+1)$$

and

$$\begin{aligned}\langle(\lambda'\mu')K'L;(20)2\|(\lambda\mu)KL\rangle &= F_{L'L}(K')\langle L'K'21|LK\rangle\delta_{KK'+1}[(1+\delta_{K'0})]^{1/2}\\ &\quad+(-1)^{\lambda'-\lambda-1}F_{L'L}(-K')\langle L'K'2-1|LK\rangle\delta_{KK'-1}[(1+\delta_{K0})]^{1/2}\end{aligned} \tag{45}$$

where the factors $F_{L'L}(K')$ are given in tables 6 and 7.

**Table 6.** The $F_{L'L}(K')$ for $(\lambda'\mu')=(\lambda, \mu-1)$.

| $L$ | $F_{L'L}(K')$ |
|---|---|
| $L'+2$ | $(\lambda-L'+K')$ |
| $L'+1$ | $\frac{1}{(L'-2K')}\{L'(\lambda+2)-2K'(\lambda+1-L')-2K'^2\}$ |
| $L'$ | $\frac{1}{(2K'+1)}\{\lambda(2K'+1)+2(K'+1)^2-\frac{2}{3}L'(L'+1)\}$ |
| $L'-1$ | $\frac{1}{(L'+2K'+1)}\{(L'+1)(\lambda+2)+2K'(\lambda+2+L')+2K'^2\}$ |
| $L'-2$ | $(\lambda+L'+K'+1)$ |

**Table 7.** The $F_{L'L}(K')$ for $(\lambda'\mu') = (\lambda-1, \mu+1)$.

| | $F_{L'L}(K') = (\mu+1-K')f_{L'L}(K')$ |
|---|---|
| $L$ | $f_{L'L}(K')$ |
| $L'+2$ | $(\lambda+\mu+1-L'+K')$ |
| $L'+1$ | $\frac{1}{(L'-2K')}\{(\lambda+\mu+1)(L'-2K')+2(L'-K')(K'+1)\}$ |
| $L'$ | $\frac{1}{(2K'+1)}\{(\lambda+\mu+1)(2K'+1)+2(K'+1)^2-\frac{2}{3}L'(L'+1)\}$ |
| $L'-1$ | $\frac{1}{(L'+2K'+1)}\{(\lambda+\mu+1)(L'+2K'+1)+2(L'+K'+1)(K'+1)\}$ |
| $L'-2$ | $(\lambda+\mu+2+L'+K')$ |

The normalisation factors $N$ now present a somewhat more challenging problem. They can be calculated via the technique of section 2. An alternative method involves the direct calculation of specific $SU(3) \supset SO(3)$ Wigner coefficients for which $L'$, $L$ and $l$ are the unique states, such as $L=0$ or $L'=1$. For these the specific state constructions of Hecht and Suzuki (1983) can be used together with simple Bargmann space integrations to evaluate some simple starting coefficients. As a specific example, with $\lambda = 2n = \text{even}$, $\mu = 2m = \text{even}$ (in the notation of Hecht and Suzuki (1983)):

$$\begin{aligned}[P^{(2n0)}(\boldsymbol{K}_1) \times P^{(2m0)}(\boldsymbol{K}_2)]^{(2n-2m,2m)}_{L=0}(\boldsymbol{K}_1 \cdot \boldsymbol{K}_1) \\ = \sqrt{3}! \sum_{(\lambda\mu)} \langle (2n-2m, 2m)L=0; (20)l=0 \| (\lambda\mu)L=0\rangle \\ \times U((20)(2n,0)(\lambda\mu)(2m,0); (2n+2,0)(2n-2m,2m)) \\ \times \left(\frac{(2n+2)!}{(2n)!2!}\right)^{1/2} [P^{(2n+2,0)}(\boldsymbol{K}_1) \times P^{(2m0)}(\boldsymbol{K}_2)]^{(\lambda\mu)}_{L=0}. \end{aligned} \tag{46}$$

Using (26) of Hecht and Suzuki (1983) for the SU(3)-coupled $\boldsymbol{K}$-space polynomials and the well known Racah coefficient we get

$$\begin{aligned}\langle (\lambda\mu)L'=0; (20)l=0 \| (\lambda+2, \mu)L=0\rangle \\ = \langle (2n-2m, 2m)0; (20)0 \| (2n+2-2m, 2m)0\rangle \\ = \left(\frac{(2n+3)(2n+2-2m)}{3(2n+1-2m)(2n+2)}\right)^{1/2} = \left(\frac{(\lambda+\mu+3)(\lambda+2)}{3(\lambda+1)(\lambda+\mu+2)}\right)^{1/2}. \end{aligned} \tag{47}$$

## Appendix

The evaluation of the $SU(3) \supset SO(3)$ Wigner coefficients is now straightforward, but it does depend on a knowledge of the matrix elements of $\mathcal{K}^{-1}$ and $\mathcal{K}$. The matrix $(\mathcal{K}\mathcal{K}^\dagger)$ is evaluated easily by the techniques of VCS theory (see in particular (17) and (18) of Rowe *et al* (1989)). The process of finding the matrix elements of $\mathcal{K}$ and $\mathcal{K}^{-1}$ involves the diagonalisation of the real Hermitian matrix $(\mathcal{K}\mathcal{K}^\dagger)$ via a unitary matrix $U$:

$$(\mathcal{K}\mathcal{K}^\dagger) = U^\dagger \lambda U \tag{A1}$$

where $\lambda = \lambda_i \delta_{ij}$ is a real positive semidefinite matrix. Note that zero eigenvalues of $\lambda$ immediately signal the occurrence of forbidden states. Since $(\mathcal{K}\mathcal{K}^\dagger)$ is diagonal in $(\lambda\mu)$ and $L$, the full $(\mathcal{K}\mathcal{K}^\dagger)$ matrix factors into submatrices whose dimension is given by the number of possible $K$ values for a particular $(\lambda\mu)L$ ($K = \mu$, $\mu - 2, \ldots 0(1)$ for $\mu =$ even (or odd)). If $\lambda_i$ denotes a *non-zero* eigenvalue, (A1) can be solved for $\mathcal{K}$ and inverted to yield

$$(\mathcal{K})_{Ki} = (U^\dagger)_{Ki}(\lambda_i)^{1/2} \qquad (\mathcal{K}^{-1})_{iK} = \frac{1}{(\lambda_i)^{1/2}} U_{iK}. \tag{A2}$$

Although the $(\mathcal{K}\mathcal{K}^\dagger)$ matrix can be evaluated numerically for any $(\lambda\mu)L$ value, it will be very useful to have analytical expressions for some of the simpler cases, which lead to one- and two-dimensional $(\mathcal{K}\mathcal{K}^\dagger)$ submatrices. These have been evaluated with the use of (17) and (18) of Rowe *et al* (1989) and will be enumerated here. Due to the central role played by the $(\mathcal{K}\mathcal{K}^\dagger)$ matrices a sketch of the method of calculation will also be given. In the VCS method the unitary character of the realisation $\gamma(X)$, see (9), is used to gain a simple recursion formula for the $(\mathcal{K}\mathcal{K}^\dagger)$ matrix. With $X = Q_\nu$, the $\nu$th spherical component of the quadrupole generator, the unitary requirement $\gamma^\dagger(Q_\nu) = (-1)^\nu \gamma(Q_{-\nu})$ leads to the relation

$$(\mathcal{K}\mathcal{K}^\dagger)(-1)^\nu \Gamma^\dagger(Q_{-\nu}) = \Gamma(Q_\nu)(\mathcal{K}\mathcal{K}^\dagger) \tag{A3}$$

as shown by Rowe *et al* (1989).

Note that the matrix $(\mathcal{K}\mathcal{K}^\dagger((\lambda\mu)L))_{K_1K_2}$ is abbreviated by $\bar{S}^L_{K_1K_2}$ in Rowe *et al* (1989), and this notation is adopted briefly here. Equation (A3) then leads to the matrix form of (17) of Rowe *et al* (1989):

$$\sum_{K_2'} \bar{S}^{L'}_{K_1'K_2'} \langle K_2 L \| \Gamma(Q) \| K_2' L' \rangle (-1)^{L'-L} = \sum_{K_1} \langle K_1' L' \| \Gamma(Q) \| K_1 L \rangle \bar{S}^L_{K_1K_2} \tag{A4}$$

where the reduced matrix elements of $\Gamma(Q)$ are given in very explicit form through (10) of Rowe *et al* (1989). For fixed choices of $K_1'$ and $K_2$ this leads to a set of recursion relations for the matrix elements of $\bar{S}^{L'}$ in terms of known matrix elements of $\bar{S}^L$. A second recursion relation, which is needed only occasionally, follows from the matrix form of (18) of Rowe *et al* (1989):

$$\sum_{K_2'K''} \bar{S}^{L'}_{K_1'K_2'} \langle K'' L' \| \Gamma(Q) \| K_2' L' \rangle \langle K_2 L \| \Gamma(Q) \| K'' L' \rangle (-1)^{L'-L}$$
$$= \sum_{K_1K''} \langle K_1' L' \| \Gamma(Q) \| K'' L' \rangle \langle K'' L' \| \Gamma(Q) \| K_1 L \rangle \bar{S}_{K_1K_2}. \tag{A5}$$

The representations $(\lambda 0)$, $(\lambda 1)$ lead to one-dimensional $\bar{S}^L$ matrices, since $K = 0$ for $(\lambda 0)$ and $K = 1$ for $(\lambda 1)$ are unique. Equation (A4) leads to the simple recursion relation for $(\lambda\mu) = (\lambda 0)$:

$$\frac{\bar{S}^{L+2}_{00}}{\bar{S}^L_{00}} = \frac{(\lambda - L)}{(\lambda + L + 3)}. \tag{A6}$$

For $\lambda =$ even, the minimum $L$ value is $L_{\min} = 0$, whereas for $\lambda =$ odd, $L_{\min} = 1$. The states with $L_{\min}$ are the starting states for the VCS construction for which $\bar{S}^{L_{\min}}_{00} = 1$. With these starting values, iteration of (A6) leads to the final results, enumerated in (A8) and (A9) below.

A somewhat more challenging example is illustrated by the representation $(\lambda 2)$ with $\lambda =$ even, and possible $L$ values:

$$\begin{array}{llllllllll} 0\ 2 & 4 & 6 & & \ldots & (\lambda-2) & & \lambda & & \\ & 2\ 3\ 4\ 5\ 6\ 7 & & & \ldots & (\lambda-2) & (\lambda-1) & \lambda & (\lambda+1) & (\lambda+2). \end{array}$$

Note that the odd-$L$ states lead to one-dimensional $\bar{S}^L$ with a unique $K$ value, $K=2$, whereas the even-$L$ states lead to two-dimensional $\bar{S}^L$ with $K=0$ and 2. Equation (A4) with the choices $K_1'=0$ and 2 and (A5) with $K_1'=0$ lead, together with $\bar{S}^0_{00}=1$, to the relations

$$\begin{aligned} &\bar{S}^2_{00}(2\lambda+8)+\bar{S}^2_{02}4\sqrt{3}=(2\lambda+2)\\ &\bar{S}^2_{20}(2\lambda+8)+\bar{S}^2_{22}4\sqrt{3}=2\sqrt{3}\\ &\bar{S}^2_{00}(4\lambda^2+26\lambda+16)-\bar{S}^2_{02}4\sqrt{3}(4\lambda+13)=(4\lambda^2+14\lambda-14). \end{aligned} \tag{A7}$$

These determine the three independent matrix elements $\bar{S}^2_{00}$, $\bar{S}^2_{22}$, $\bar{S}^2_{02}=\bar{S}^2_{20}$. With these known matrix elements, (A4) can then be used to determine $\bar{S}^3_{22}$ and, with $L'=L+2$, leads to a recursive determination of $\bar{S}^L_{22}$ for odd $L$ values. Using these and the known matrix elements of $\bar{S}^2$, (A4) can then be used recursively to determine $\bar{S}^{L'}_{00}$, $\bar{S}^{L'}_{22}$, $\bar{S}^{L'}_{02}=\bar{S}^{L'}_{20}$ with $L'=$ even from known matrix elements of $\bar{S}^{L'-1}$ and $\bar{S}^{L'-2}$, the latter with $K$ values of 0 and 2. This recursive process leads to the results enumerated in (A14) and (A15).

For $(\lambda 0)$ $\lambda =$ even:

$$(\mathscr{K}(\lambda 0)L)=\left(\frac{(\lambda+1)!!\,\lambda!!}{(\lambda+L+1)!!(\lambda-L)!!}\right)^{1/2}. \tag{A8}$$

For $(\lambda 0)$ $\lambda =$ odd:

$$(\mathscr{K}(\lambda 0)L)=\left(\frac{(\lambda+2)!!(\lambda-1)!!}{(\lambda+L+1)!!(\lambda-L)!!}\right)^{1/2} \tag{A9}$$

where $a!!=a(a-2)(a-4)\ldots$. Note that these $\mathscr{K}$ are normalised such that $\mathscr{K}(\lambda 0)L=0)=1$ for even $\lambda$ and $\mathscr{K}(\lambda 0)L=1)=1$ for odd $\lambda$.

For $(\lambda 1)$ $\lambda =$ even, $L=$ even:

$$(\mathscr{K}(\lambda 1)L)=\left(\frac{\lambda!!(\lambda+3)!!}{(\lambda+2)(\lambda-L)!!(\lambda+L+1)!!}\right)^{1/2}. \tag{A10}$$

For $(\lambda 1)$ $\lambda =$ even, $L=$ odd:

$$(\mathscr{K}(\lambda 1)L)=\left(\frac{\lambda!!(\lambda+3)!!}{(\lambda+1-L)!!(\lambda+L+2)!!}\right)^{1/2}. \tag{A11}$$

For $(\lambda 1)$ $\lambda =$ odd, $L=$ even:

$$(\mathscr{K}(\lambda 1)L)=\left(\frac{(\lambda+2)(\lambda-1)!!(\lambda+2)!!}{(\lambda+1-L)!!(\lambda+L+2)!!}\right)^{1/2}. \tag{A12}$$

For $(\lambda 1)\lambda =$ odd, $L=$ odd:

$$(\mathscr{K}(\lambda 1)L)=\left(\frac{(\lambda-1)!!(\lambda+2)!!}{(\lambda-L)!!(\lambda+L+1)!!}\right)^{1/2}. \tag{A13}$$

For $(\lambda 2)$ with $\lambda$ = even, $L$ = even defining a common factor (CF):

$$(\mathrm{CF}) = \frac{\lambda(\lambda-2)!!(\lambda+5)!!}{2(\lambda+2)(\lambda+5)(\lambda+2-L)!!(\lambda+L+3)!!} \tag{A14a}$$

$$(\mathcal{K}\mathcal{K}^\dagger((\lambda 2)L))_{22} = (\mathrm{CF})\tfrac{1}{2}[2(\lambda+3)^2 - L(L+1)] \tag{A14b}$$

$$(\mathcal{K}\mathcal{K}^\dagger((\lambda 2)L))_{00} = (\mathrm{CF})[2(\lambda+2)^2 - L(L+1)] \tag{A14c}$$

$$(\mathcal{K}\mathcal{K}^\dagger((\lambda 2)L))_{02} = (\mathrm{CF})[\tfrac{1}{2}(L-1)L(L+1)(L+2)]^{1/2}. \tag{A14d}$$

Note that with $L = \lambda + 2$ this $2 \times 2$ matrix $(\mathcal{K}\mathcal{K}^\dagger)$ has the form

$$(\mathrm{CF})^2 \begin{pmatrix} \tfrac{1}{2}(\lambda+3)(\lambda+4) & [\tfrac{1}{2}(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)]^{1/2} \\ [\tfrac{1}{2}(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)]^{1/2} & (\lambda+1)(\lambda+2) \end{pmatrix}.$$

It can be seen at once that this matrix has one zero eigenvalue, signalling the fact that there is only one allowed state with $L = \lambda + 2$, corresponding to the non-zero eigenvalue.

For $(\lambda 2)$ with $\lambda$ = even, $L$ = odd

$$(\mathcal{K}\mathcal{K}^\dagger((\lambda 2)L))_{22} = \frac{\lambda(\lambda+3)(\lambda-2)!!(\lambda+5)!!}{2(\lambda+2)(\lambda+5)(\lambda+1-L)!!(\lambda+2+L)!!}. \tag{A15}$$

For $(\lambda 2)$ with $\lambda$ = odd $L$ = odd, using the common factor

$$(\mathrm{CF}) = \frac{(\lambda-1)!!(\lambda+4)!!}{2(\lambda+3)(\lambda+2-L)!!(\lambda+L+3)!!} \tag{A16a}$$

$$(\mathcal{K}\mathcal{K}^\dagger((\lambda 2)L))_{22} = (\mathrm{CF})\tfrac{1}{2}[2(\lambda+3)^2 - L(L+1)] \tag{A16b}$$

$$(\mathcal{K}\mathcal{K}^\dagger((\lambda 2)L))_{00} = (\mathrm{CF})[2(\lambda+2)^2 - L(L+1)] \tag{A16c}$$

$$(\mathcal{K}\mathcal{K}^\dagger((\lambda 2)L))_{02} = (\mathrm{CF})[\tfrac{1}{2}(L-1)L(L+1)(L+2)]^{1/2}. \tag{A16d}$$

Note again that with $L = \lambda + 2$ there is one zero eigenvalue.

For $(\lambda 2)$ with $\lambda$ = odd, $L$ = even:

$$(\mathcal{K}\mathcal{K}^\dagger((\lambda 2)L))_{22} = \frac{(\lambda-1)!!(\lambda+4)!!}{2(\lambda+1-L)!!(\lambda+2+L)!!}. \tag{A17}$$

Finally for states with $(\lambda\mu) = (\lambda 3)$ the $(\mathcal{K}\mathcal{K}^\dagger)$ matrices are as follows.

For $\lambda$ = even, with $L$ = even:

$$(\mathrm{CF}) = \frac{\lambda!!(\lambda+5)!!}{4(\lambda+2)(\lambda+4)(\lambda+3+L)!!(\lambda+2-L)!!} \tag{A18a}$$

while for $\lambda$ = odd, with $L$ = odd:

$$(\mathrm{CF}) = \frac{(\lambda-1)!!(\lambda+4)!!}{4(\lambda+3)(\lambda+3+L)!!(\lambda+2-L)!!} \tag{A18b}$$

and for these two cases

$$(\mathcal{K}\mathcal{K}^\dagger((\lambda 3)L))_{33} = (\mathrm{CF})\tfrac{1}{3}[4\lambda^2 + 32\lambda + 66 - L(L+1)] \tag{A18c}$$

$$(\mathcal{K}\mathcal{K}^\dagger((\lambda 3)L))_{11} = (\mathrm{CF})[4\lambda^2 + 16\lambda + 18 - 3L(L+1)] \tag{A18d}$$

$$(\mathcal{K}\mathcal{K}^\dagger((\lambda 3)L))_{13} = (\mathrm{CF})[(L-2)(L-1)(L+2)(L+3)]^{1/2}. \tag{A18e}$$

Note that there is one zero eigenvalue for $L = \lambda + 2$.

For $(\lambda\mu)=(\lambda 3)$, but with $\lambda$ = even, $L$ = odd:

$$(\text{CF})=\frac{\lambda!!(\lambda+5)!!}{4(\lambda+2)(\lambda+L+4)!!(\lambda+3-L)!!} \tag{A19a}$$

while for $\lambda$ = odd, $L$ = even:

$$(\text{CF})=\frac{(\lambda+4)(\lambda+4)!!(\lambda-1)!!}{4(\lambda+3)(\lambda+L+4)!!(\lambda+3-L)!!} \tag{A19b}$$

and for these two cases:

$$(\mathcal{K}\mathcal{K}^{\dagger}((\lambda 3)L))_{33}=(\text{CF})\tfrac{1}{3}[4\lambda^2+32\lambda+66-3L(L+1)] \tag{A19c}$$

$$(\mathcal{K}\mathcal{K}^{\dagger}((\lambda 3)L))_{11}=(\text{CF})[4\lambda^2+16\lambda+18-L(L+1)] \tag{A19d}$$

$$(\mathcal{K}\mathcal{K}^{\dagger}((\lambda 3)L))_{13}=(\text{CF})[(L-2)(L-1)(L+2)(L+3)]^{1/2}. \tag{A19e}$$

Note that there is one zero eigenvalue for $L=\lambda+3$.

For representations $(\lambda\mu)$ with $\mu\geqslant 4$, the $(\mathcal{K}\mathcal{K}^{\dagger})$ submatrices have dimensions $\geqslant 3$ and it may be best to evaluate matrix elements numerically. However, analytic expressions can also be obtained. All the ingredients needed to evaluate the $SU(3)\supset SO(3)$ Wigner coefficients can therefore be made available in relatively simple analytic form. Only the diagonalisation of the $(\mathcal{K}\mathcal{K}^{\dagger})$ matrices and the evaluation of the $U_{iK}$ are left to be done numerically in specific cases.

Finally, some of the $\mathcal{K}\mathcal{K}^{\dagger}$ submatrices for $L=2$, arbitrary $(\lambda\mu)$, are $1\times 1$ matrices, with uniquely determined values of $K$. These are often needed as stepping stones in other calculations. They have simple values.

For $\lambda$ = even, $\mu$ = odd:

$$(\mathcal{K}((\lambda\mu)L=2))_{11}=\left(\frac{\lambda}{\lambda+2}\right)^{1/2}. \tag{A20}$$

For $\lambda$ = odd, $\mu$ = odd:

$$(\mathcal{K}((\lambda\mu)L=2))_{11}=\left(\frac{\lambda+\mu+1}{\lambda+\mu+3}\right)^{1/2}. \tag{A21}$$

For $\lambda$ = odd, $\mu$ = even:

$$(\mathcal{K}((\lambda\mu)L=2))_{22}=\left(\frac{\mu}{\mu+2}\right)^{1/2}. \tag{A22}$$

## References

Deenen J and Quesne C 1984 *J. Math. Phys.* **25** 1638, 2354
Draayer J P and Akiyama Y 1973a *J. Math. Phys.* **14** 1904
—— 1973b *Comput. Phys. Commun.* **5** 405
Elliott J P 1958 *Proc. R. Soc.* A **245** 128, 562
Hecht K T 1987 *The Vector Coherent State Method and its Application to Problems of Higher Symmetries* (*Lecture Notes in Phys.* **290**) (Berlin: Springer)
—— 1989 *Nucl. Phys.* A **493** 29
Hecht K T and Suzuki Y 1983 *J. Math. Phys.* **24** 785
LeBlanc R and Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** 31

LeBlanc R and Rowe D J 1985 *J. Phys. A: Math. Gen.* **18** 1891, 1905
—— 1986 *J. Phys. A: Math. Gen.* **19** 1083
Rowe D J 1984 *J. Math. Phys.* **25** 2662
Rowe D J, LeBlanc R and Hecht K T 1988 *J. Math. Phys.* **29** 287
Rowe D J, LeBlanc R and Repka J 1989 *J. Phys. A: Math. Gen.* **22** L309
Vergados J D 1968 *Nucl. Phys.* A **111** 681